Introduction to Mathematics and Modeling

lecture 3
Second order differential equations

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lecture : 3
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This week

1. Section 17.1: Second-order linear differential equations
2. Section 17.2: Inhomogeneous linear equations
3. Section 17.3: Applications
A second-order linear differential equations with constant coefficients is a differential equation of the form

\[ ay'' + by' + cy = g(x) \]

where \( a, b \) and \( c \) are real constants and where \( g(x) \) is a function.

- **Second-order**: \( y'' \) appears in the equation.
- **Linear**: no nonlinear terms of \( y, y', y'' \).
- **Constant coefficients**: \( a, b, c \) are independent of \( x \)
- If \( g(x) = 0 \) then the equation is **homogeneous**.
- If \( g(x) \neq 0 \) then the equation is **inhomogeneous**. The function \( g(x) \) is called the **input** or **forcing term**.
Homogeneous second-order linear differential equations

\[ ay'' + by' + cy = 0 \]

**Superposition principle**

*If* \( y_1(x) \) *and* \( y_2(x) \) *are solutions of a homogeneous second-order linear differential equation, then*

\[ y(x) = k_1 y_1(x) + k_2 y_2(x) \]

*is also a solution for all constants* \( k_1 \) *and* \( k_2 \).

⚠️ The superposition principle does not hold for inhomogeneous differential equations.
Homogeneous second-order linear differential equations

$$ay'' + by' + cy = 0$$

- Two solutions $y_1(x)$ and $y_2(x)$ are considered to be **different** if one is not a scaled version of the other.

**Theorem**

If $y_1(x)$ and $y_2(x)$ are different solutions of a homogeneous second-order linear differential equation, then all solutions can be described by

$$y(x) = k_1 y_1(x) + k_2 y_2(x).$$

(*)

where $k_1$ and $k_2$ are arbitrary constants.

- The formula $k_1 y_1(x) + k_2 y_2(x)$ is called the **general solution**.
Trial solution \( y(x) = e^{\lambda x} \)

\[
ay'' + by' + cy = 0
\]

- Differentiate \( y(x) \):

\[
\begin{align*}
y(x) &= e^{\lambda x} \\
y'(x) &= \lambda e^{\lambda x} \\
y''(x) &= \lambda^2 e^{\lambda x}
\end{align*}
\]
Trial solution \( y(x) = e^{\lambda x} \)

\[
ay'' + by' + cy = 0
\]

- Differentiate \( y(x) \):
  
  \[
  y(x) = e^{\lambda x} \\
  y'(x) = \lambda e^{\lambda x} \\
  y''(x) = \lambda^2 e^{\lambda x}
  \]

- Substitution in the differential equation gives
  
  \[
a\lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + c e^{\lambda x} = 0 \quad \Rightarrow \quad e^{\lambda x} (a\lambda^2 + b\lambda + c) = 0
  \]

  \[
  \Rightarrow a\lambda^2 + b\lambda + c = 0
  \]
Trial solution \( y(x) = e^{\lambda x} \)

\[
ay'' + by' + cy = 0
\]  \hspace{1cm} (1)

- If \( y(x) = e^{\lambda x} \) is a solution of (1), then \( a\lambda^2 + b\lambda + c = 0 \).
Trial solution \( y(x) = e^{\lambda x} \)

\[
ay'' + by' + cy = 0
\]  
(1)

- If \( y(x) = e^{\lambda x} \) is a solution of (1), then \( a\lambda^2 + b\lambda + c = 0 \).
- The zeros of equation (2) can be found with the **quadratic formula**:
Trial solution $y(x) = e^{\lambda x}$

\[
ay'' + by' + cy = 0
\]

(1)

- If $y(x) = e^{\lambda x}$ is a solution of (1), then $a \lambda^2 + b \lambda + c = 0$.
- The zeros of equation (2) can be found with the **quadratic formula**:

**Theorem**

*If $b^2 - 4ac > 0$ then the general solution of (1) is*

\[ y(x) = k_1 e^{\lambda_1 x} + k_2 e^{\lambda_2 x}, \]

*where*

\[ \lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \]
Trial solution \( y(x) = e^{\lambda x} \)

\[
ay'' + by' + cy = 0
\]  \( (1) \)

- If \( y(x) = e^{\lambda x} \) is a solution of (1), then \( a\lambda^2 + b\lambda + c = 0 \).
- The zeros of equation (2) can be found with the quadratic formula:

Theorem

If \( b^2 - 4ac > 0 \) then the general solution of (1) is

\[ y(x) = k_1 e^{\lambda_1 x} + k_2 e^{\lambda_2 x}, \]

where

\[ \lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \]

- If \( b^2 - 4ac \leq 0 \), then (1) does not have two different solutions of the form \( e^{\lambda x} \).
Trial solution \( y(x) = e^{\sigma x} \cos(\omega x) \) with \( \omega \neq 0 \)

\[
ay'' + by' + cy = 0
\]

- Differentiate \( y(x) \):

\[
y(x) = e^{\sigma x} \cos(\omega x)
\]

\[
y'(x) = \sigma e^{\sigma x} \cos(\omega x) - \omega e^{\sigma x} \sin(\omega x)
\]

\[
y''(x) = \sigma^2 e^{\sigma x} \cos(\omega x) - 2\sigma \omega e^{\sigma x} \sin(\omega x) - \omega^2 e^{\sigma x} \cos(\omega x)
\]
Trial solution $y(x) = e^{\sigma x} \cos(\omega x)$ with $\omega \neq 0$

$ay'' + by' + cy = 0$

- Differentiate $y(x)$:
  
  $y(x) = e^{\sigma x} \cos(\omega x)$
  
  $y'(x) = \sigma e^{\sigma x} \cos(\omega x) - \omega e^{\sigma x} \sin(\omega x)$
  
  $y''(x) = \sigma^2 e^{\sigma x} \cos(\omega x) - 2\sigma \omega e^{\sigma x} \sin(\omega x) - \omega^2 e^{\sigma x} \cos(\omega x)$

- Substitution in the differential equation gives
  
  $(a(\sigma^2 - \omega^2) + b\sigma + c)e^{\sigma x} \cos(\omega x) - \omega(2a\sigma + b)e^{\sigma x} \sin(\omega x) = 0.$
Trial solution $y(x) = e^{\sigma x} \cos(\omega x)$ with $\omega \neq 0$

\[
ay'' + by' + cy = 0
\]

- Differentiate $y(x)$:
  
  \[
y(x) = e^{\sigma x} \cos(\omega x)
  \]
  
  \[
y'(x) = \sigma e^{\sigma x} \cos(\omega x) - \omega e^{\sigma x} \sin(\omega x)
  \]
  
  \[
y''(x) = \sigma^2 e^{\sigma x} \cos(\omega x) - 2\sigma \omega e^{\sigma x} \sin(\omega x) - \omega^2 e^{\sigma x} \cos(\omega x)
  \]

- Substitution in the differential equation gives
  
  \[
  (a(\sigma^2 - \omega^2) + b\sigma + c)e^{\sigma x} \cos(\omega x) - \omega(2a\sigma + b)e^{\sigma x} \sin(\omega x) = 0.
  \]

- This implies
  
  \[
a(\sigma^2 - \omega^2) + b\sigma + c = 0 \quad \text{and} \quad \omega(2a\sigma + b) = 0
  \]
Trial solution $y(x) = e^{\sigma x} \cos(\omega x)$ with $\omega \neq 0$

- The function $y(x) = e^{\sigma x} \cos(\omega x)$ is a solution if and only if

$$\begin{cases} a(\sigma^2 - \omega^2) + b\sigma + c = 0 \\ \omega(2a\sigma + b) = 0 \end{cases}$$
The function \( y(x) = e^{\sigma x} \cos(\omega x) \) is a solution if and only if

\[
\begin{align*}
    a(\sigma^2 - \omega^2) + b\sigma + c &= 0 \\
    \omega(2a\sigma + b) &= 0
\end{align*}
\]

Since \( \omega \neq 0 \) we have \( \sigma = -b/2a \), but then

\[
\omega^2 = \frac{-b^2 + 4ac}{4a^2} \quad \Rightarrow \quad \omega = \pm \frac{\sqrt{-\left(b^2 - 4ac\right)}}{2a}
\]

This precisely works when \( b^2 - 4ac < 0 \)!
Trial solution $y(x) = e^{\sigma x} \cos(\omega x)$ with $\omega \neq 0$

- The function $y(x) = e^{\sigma x} \cos(\omega x)$ is a solution if and only if

\[
\begin{aligned}
\begin{cases}
 a(\sigma^2 - \omega^2) + b\sigma + c = 0 \\
 \omega(2a\sigma + b) = 0
\end{cases}
\end{aligned}
\]

- Since $\omega \neq 0$ we have $\sigma = -b/2a$, but then

\[
\begin{aligned}
 \omega^2 &= \frac{-b^2 + 4ac}{4a^2} \\
 \omega &= \pm \frac{\sqrt{- (b^2 - 4ac)}}{2a}
\end{aligned}
\]

This precisely works when $b^2 - 4ac < 0$!

- Notice that

\[
\lambda = \sigma \pm \omega i = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

are the (complex) roots of the equation $a\lambda^2 + b\lambda + c = 0$. 
The function \( y(x) = e^{\sigma x} \sin(\omega x) \) is a solution of \( ay'' + by' + cy = 0 \) if and only if
\[
\begin{align*}
    a(\sigma^2 - \omega^2) + b\sigma + c &= 0 \\
    \omega(2a\sigma + b) &= 0
\end{align*}
\]
The function \( y(x) = e^{\sigma x} \sin(\omega x) \) is a solution of \( ay'' + by' + cy = 0 \) if and only if
\[
\begin{align*}
\left\{ \begin{array}{l}
 a(\sigma^2 - \omega^2) + b\sigma + c = 0 \\
 \omega(2a\sigma + b) = 0
\end{array} \right.
\]
These are the same conditions as for \( y(x) = e^{\sigma x} \cos(\omega x) \):

If \( y(x) = e^{\sigma x} \sin(\omega x) \) is a solution then \( y(x) = e^{\sigma x} \cos(\omega x) \) is a solution, and vice versa.
The function $y(x) = e^{\sigma x} \sin(\omega x)$ is a solution of $ay'' + by' + cy = 0$ if and only if

$$\begin{cases}
a(\sigma^2 - \omega^2) + b\sigma + c = 0 \\
\omega(2a\sigma + b) = 0
\end{cases}$$

These are the same conditions as for $y(x) = e^{\sigma x} \cos(\omega x)$:

If $y(x) = e^{\sigma x} \sin(\omega x)$ is a solution then $y(x) = e^{\sigma x} \cos(\omega x)$ is a solution, and vice versa.

**Theorem**

If $b^2 - 4ac < 0$ then the general solution of $ay'' + by' + cy = 0$ is

$$y(x) = e^{\sigma x} \left[ k_1 \cos(\omega x) + k_2 \sin(\omega x) \right],$$

where

$$\sigma = -\frac{b}{2a} \quad \text{and} \quad \omega = \frac{\sqrt{-b^2 + 4ac}}{2a}.$$
1. Consider the differential equation

\[ y'' + cy = 0. \]  

(1)

(a) For which values of \( c \) is \( y(x) = \sin(\omega x) \) (with \( \omega > 0 \)) a solution?

(b) Express \( \omega \) as a function of \( c \).

(c) For the values of \( c \) found in (a), show that \( y(x) = \cos(\omega x) \) (with \( \omega \) as found in (b)) is also a solution of the differential equation.

(d) Find the general solution to (1).

2. Consider the differential equation

\[ y'' + by' + y = 0. \]  

(2)

(a) For which value of \( b \) is \( y(x) = x e^x \) a solution?

(b) For the value of \( b \) found in (a), find the general solution to (2).
Definition

Let $ay'' + by' + cy = 0$ be a homogeneous linear differential equation. The quadratic equation

$$a\lambda^2 + b\lambda + c = 0.$$ 

is called the auxiliary equation or characteristic equation.

- The roots of the auxiliary equation are determined by the quadratic formula:

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$ 

- The number $b^2 - 4ac$ is called the discriminant.

- There are three possibilities:

  - **Case 1**: $b^2 - 4ac > 0$,
  - **Case 2**: $b^2 - 4ac < 0$,
  - **Case 3**: $b^2 - 4ac = 0$. 
Solution method for Case 1 (positive discriminant)

\[ a\lambda^2 + b\lambda + c = 0 \implies \lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

**Case 1:** \[ b^2 - 4ac > 0 \]

\[ \lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \]

General solution:

\[
\begin{align*}
    y_1(x) &= e^{\lambda_1 x} \\
    y_2(x) &= e^{\lambda_2 x}
\end{align*}
\] \[ \implies y(x) = k_1 e^{\lambda_1 x} + k_2 e^{\lambda_2 x} \]
Solution method for Case 2 (negative discriminant)

\[ a\lambda^2 + b\lambda + c = 0 \quad \Rightarrow \quad \lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

**Case 2:** \( b^2 - 4ac < 0 \)

\[ \lambda = \sigma \pm i\omega \quad \Rightarrow \quad \sigma = -\frac{b}{2a} \quad \text{and} \quad \omega = \frac{\sqrt{-b^2 + 4ac}}{2a} \]

**General solution:**

\[
\begin{align*}
    y_1(x) &= e^{\sigma x} \sin(\omega x) \\
    y_2(x) &= e^{\sigma x} \cos(\omega x)
\end{align*}
\]

\[ \Rightarrow \quad y(x) = k_1 e^{\sigma x} \sin(\omega x) + k_2 e^{\sigma x} \cos(\omega x) \]
Solution method for Case 3 (zero discriminant)

\[ a\lambda^2 + b\lambda + c = 0 \quad \Rightarrow \quad \lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

Case 3: \[ b^2 - 4ac = 0 \]

\[ \lambda = -\frac{b}{2a} \]

General solution:

\[
\begin{align*}
y_1(x) &= e^{\lambda x} \\
y_2(x) &= xe^{\lambda x}
\end{align*}
\]

\[ \Rightarrow \quad y(x) = k_1 e^{\lambda x} + k_2 xe^{\lambda x} \]
Example 1

\[ y'' - y' - 6y = 0 \]

- Solve the characteristic equation \((a = 1, b = -1, c = -6)\)

\[
\lambda^2 - \lambda - 6 = 0 \quad \Rightarrow \quad \lambda = \frac{1 \pm \sqrt{25}}{2} \quad \Rightarrow \quad \lambda = 3 \text{ and } \lambda = -2
\]
Example 1

\[ y'' - y' - 6y = 0 \]

- Solve the characteristic equation \((a = 1, b = -1, c = -6)\)

\[ \lambda^2 - \lambda - 6 = 0 \quad \Rightarrow \quad \lambda = \frac{1 \pm \sqrt{25}}{2} \quad \Rightarrow \quad \lambda = 3 \text{ and } \lambda = -2 \]

- Case 1 applies: the general solution is

\[ y(x) = k_1 e^{3x} + k_2 e^{-2x} \]
Example 2

\[ y'' - 4y' + 5y = 0 \]

- Solve the **characteristic equation** \((a = 1, b = -4, c = 5)\)

\[ \lambda^2 - 4\lambda + 5 = 0 \quad \Rightarrow \quad \lambda = \frac{4 \pm \sqrt{-4}}{2} \]
Example 2

\[ y'' - 4y' + 5y = 0 \]

- Solve the **characteristic equation** \((a = 1, b = -4, c = 5)\)

\[ \lambda^2 - 4\lambda + 5 = 0 \quad \Rightarrow \quad \lambda = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm i\sqrt{4}}{2} = 2 \pm i \]

- Case 2 applies: find \(\sigma\) and \(\omega\):

\[ \sigma = 2 \quad \text{and} \quad \omega = 1 \]
Example 2

$$y'' - 4y' + 5y = 0$$

- Solve the characteristic equation \((a = 1, b = -4, c = 5)\)

\[
\lambda^2 - 4\lambda + 5 = 0 \quad \Rightarrow \quad \lambda = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm i\sqrt{4}}{2} = 2 \pm i
\]

- Case 2 applies: find \(\sigma\) and \(\omega\):

\[
\sigma = 2 \quad \text{and} \quad \omega = 1
\]

- The general solution is

\[
y(x) = k_1 e^{2x} \cos(x) + k_2 e^{2x} \sin(x)
\]
Example 3

\[ y'' + 2y' + y = 0 \]

- Solve the characteristic equation \((a = 1, b = 2, c = 1)\)

\[ \lambda^2 + 2\lambda + 1 = 0 \quad \Rightarrow \quad \lambda = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 1}}{2} \]

Case 3 applies: the general solution is

\[ y(x) = k_1 e^{-x} + k_2 xe^{-x} \]
Example 3

\[ y'' + 2y' + y = 0 \]

- Solve the **characteristic equation** \((a = 1, b = 2, c = 1)\)

\[ \lambda^2 + 2\lambda + 1 = 0 \quad \Rightarrow \quad \lambda = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 1}}{2} = \frac{-2 \pm \sqrt{0}}{2} = -1 \]

- Case 3 applies: the general solution is

\[ y(x) = k_1 e^{-x} + k_2 xe^{-x} \]
Assignment: IMM1 - Tutorial 3.2
If

\[ ay'' + by' + cy = g(x), \]  

is a nonhomogeneous linear differential equation, then the **complementary equation** of (1) is

\[ ay'' + by' + cy = 0. \]
If
\[ ay'' + by' + cy = g(x), \] (1)
is a nonhomogeneous linear differential equation, then the
**complementary equation of (1)** is
\[ ay'' + by' + cy = 0. \] (2)

**Theorem**
Sec. 17.2, theorem 7

The general solution to the nonhomogeneous equation (1) has the form
\[ y(x) = y_c(x) + y_p(x), \]
where the **complementary solution** \( y_c(x) \) is the general solution to the associated homogeneous equation (2), and \( y_p(x) \) is any solution to the nonhomogeneous equation (1).
Inhomogeneous linear equations

If
\[ ay'' + by' + cy = g(x), \]  
(1)
is a nonhomogeneous linear differential equation, then the **complementary equation of** (1) is
\[ ay'' + by' + cy = 0. \]  
(2)

**Theorem**

The general solution to the nonhomogeneous equation (1) has the form
\[ y(x) = y_c(x) + y_p(x), \]
where the **complementary solution** \( y_c(x) \) is the general solution to the associated homogeneous equation (2), and \( y_p(x) \) is any solution to the nonhomogeneous equation (1).

\[ y_p(x) \] is called a **particular solution**.

⚠️ You only need one particular solution!
Finding a particular solution to (1) depends on \( g(x) \) and is often very hard. There are several methods to find particular solutions.
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- The **method of Undetermined Coefficients** is based on trial solutions which are similar to \( g(x) \) and that contain unknown constants.
Particular solutions

\[ ay'' + by' + cy = g(x), \]  

(1)

Finding a particular solution to (1) depends on \( g(x) \) and is often very hard. There are several methods to find particular solutions.

The **method of Undetermined Coefficients** is based on trial solutions which are similar to \( g(x) \) and that contain unknown constants.

For a full description, see Thomas’ Calculus section 17.2. We will study this method for the special case where \( g(x) = g_0 \) is a constant.
Particular solutions

\[ ay'' + by' + cy = g(x), \]  

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- The **method of Undetermined Coefficients** is based on trial solutions which are similar to \( g(x) \) and that contain unknown constants.

- For a full description, see Thomas’ Calculus section 17.2. We will study this method for the special case where \( g(x) = g_0 \) is a constant.

**Attempt 1**

*If \( g(x) = g_0 \), try \( y_p(x) = A \), where \( A \) is a constant.*
Example

Find the general solution to $y'' - 3y' + 2y = 6$. (1)
Example

Find the general solution to $y'' - 3y' + 2y = 6$. (1)

- The solution to the complementary equation $y'' - 3y' + 2y = 0$ is
  \[ y_h(x) = k_1 e^x + k_2 e^{2x}. \]
Example

Find the general solution to $y'' - 3y' + 2y = 6$.  \hspace{1cm} (1)

- The solution to the complementary equation $y'' - 3y' + 2y = 0$ is 
  
  \[ y_h(x) = k_1 e^x + k_2 e^{2x}. \]

- Try $y_p(x) = A$. 


Example

Find the general solution to \( y'' - 3y' + 2y = 6 \). (1)

- The solution to the complementary equation \( y'' - 3y' + 2y = 0 \) is
  \[ y_h(x) = k_1 e^x + k_2 e^{2x}. \]
- Try \( y_p(x) = A \).
- Observe that \( y_p''(x) = y_p'(x) = 0 \).
Example

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- Try \( y_p(x) = A \).
- Observe that \( y_p''(x) = y_p'(x) = 0 \).
- Substitution of \( y_p(x) = A \) in (1) gives the equation \( 2A = 6 \).
Example

Find the general solution to \( y'' - 3y' + 2y = 6 \). \hspace{1cm} (1)

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- Observe that \( y_p''(x) = y_p'(x) = 0 \).
- Substitution of \( y_p(x) = A \) in (1) gives the equation \( 2A = 6 \).
- From this equation follows \( y_p(x) = 3 \).
Example

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- The solution to the complementary equation \( y'' - 3y' + 2y = 0 \) is
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y_h(x) = k_1 e^x + k_2 e^{2x}.
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- Try \( y_p(x) = A \).
- Observe that \( y_p''(x) = y_p'(x) = 0 \).
- Substitution of \( y_p(x) = A \) in (1) gives the equation \( 2A = 6 \).
- From this equation follows \( y_p(x) = 3 \).
- The general solution to (1) is
  \[
y(x) = y_h(x) + y_p(x) = k_1 e^x + k_2 e^{2x} + 3
  \]
Example

Find the general solution to $y'' - 3y' = 6$. (1)
Example

Find the general solution to $y'' - 3y' = 6$. (1)

- The solution to the complementary equation $y'' - 3y' = 0$ is
  
  \[ y_h(x) = k_1 e^{3x} + k_2. \]
Example

Find the general solution to $y'' - 3y' = 6$. (1)

- The solution to the complementary equation $y'' - 3y' = 0$ is $y_h(x) = k_1 e^{3x} + k_2$.
- Trying $y_p(x) = A$ and substituting $y_p(x) = A$ in (1) gives the equation $0 = 6$, so the particular solution cannot be constant!
Example

Find the general solution to $y'' - 3y' = 6$. (1)

- The solution to the complementary equation $y'' - 3y' = 0$ is
  \[ y_h(x) = k_1 e^{3x} + k_2. \]
- Trying $y_p(x) = A$ and substituting $y_p(x) = A$ in (1) gives the equation $0 = 6$, so the particular solution cannot be constant!
- Try $y_p(x) = Ax$. 
Example

Find the general solution to \( y'' - 3y' = 6 \). \( (1) \)

- The solution to the complementary equation \( y'' - 3y' = 0 \) is
  \[
y_h(x) = k_1 e^{3x} + k_2.
  \]
- Trying \( y_p(x) = A \) and substituting \( y_p(x) = A \) in (1) gives the
equation \( 0 = 6 \), so the particular solution cannot be constant!
- Try \( y_p(x) = Ax \).
- Now \( y'_p(x) = A \) and \( y''_p(x) = 0 \).
Example

**Find the general solution to** \( y'' - 3y' = 6 \). \( (1) \)

- The solution to the complementary equation \( y'' - 3y' = 0 \) is
  \[
y_h(x) = k_1 e^{3x} + k_2.
  \]
- Trying \( y_p(x) = A \) and substituting \( y_p(x) = A \) in \( (1) \) gives the equation \( 0 = 6 \), so the particular solution **cannot be constant**!
- Try \( y_p(x) = Ax \).
- Now \( y'_p(x) = A \) and \( y''_p(x) = 0 \).
- Substitution of \( y_p(x) = Ax \) in \( (1) \) gives the equation \( -3A = 6 \).
Example

Find the general solution to $y'' - 3y' = 6$. (1)

- The solution to the complementary equation $y'' - 3y' = 0$ is $y_h(x) = k_1 e^{3x} + k_2$.
- Trying $y_p(x) = A$ and substituting $y_p(x) = A$ in (1) gives the equation $0 = 6$, so the particular solution cannot be constant!
- Try $y_p(x) = Ax$.
- Now $y_p'(x) = A$ and $y_p''(x) = 0$.
- Substitution of $y_p(x) = Ax$ in (1) gives the equation $-3A = 6$.
- From this equation follows $A = -2$, so $y_p(x) = -2x$. 
Find the general solution to \( y'' - 3y' = 6 \).  \((1)\)

The solution to the complementary equation \( y'' - 3y' = 0 \) is
\[
y_h(x) = k_1 e^{3x} + k_2.
\]

Trying \( y_p(x) = A \) and substituting \( y_p(x) = A \) in (1) gives the equation \( 0 = 6 \), so the particular solution cannot be constant!

Try \( y_p(x) = Ax \).

Now \( y'_p(x) = A \) and \( y''_p(x) = 0 \).

Substitution of \( y_p(x) = Ax \) in (1) gives the equation \(-3A = 6\).

From this equation follows \( A = -2 \), so \( y_p(x) = -2x \).

The general solution to (1) is
\[
y(x) = y_h(x) + y_p(x) = k_1 e^{3x} + k_2 - 2x
\]
Definition

A second order initial or boundary value problem is a second order differential equation accompanied by two conditions.
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A second order initial or boundary value problem is a second order differential equation accompanied by two conditions.

- General form of an initial value problem:

\[
\begin{aligned}
ay'' + by' + cy &= g(x), \\
y(x_0) &= y_0, \\
y'(x_0) &= y_1.
\end{aligned}
\]
**Definition**

A second order **initial** or **boundary** value problem is a second order differential equation accompanied by two conditions.

- General form of an initial value problem:

  \[
  \begin{cases}
  ay'' + by' + cy = g(x), \\
  y(x_0) = y_0, \\
  y'(x_0) = y_1.
  \end{cases}
  \tag{1}
  \]

- The conditions \( y(x_0) = y_0 \) and \( y'(x_0) = y_1 \) are called **initial conditions**.
Definition

A second order initial or boundary value problem is a second order differential equation accompanied by two conditions.

General form of an initial value problem:

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\begin{cases}
ay'' + by' + cy = g(x), \\
y(x_0) = y_0, \\
y'(x_0) = y_1.
\end{cases}
\]

The conditions \(y(x_0) = y_0\) and \(y'(x_0) = y_1\) are called initial conditions.

Theorem

There is only one function \(y(x)\) that satisfies initial value problem (1).
Example 4

\[
\begin{cases}
  y'' - 2y' + y = 0, \\
  y(0) = 1, \\
  y'(0) = -1.
\end{cases}
\]

The characteristic equation is

\[\lambda^2 - 2\lambda + 1 = 0,\]

The discriminant is \(0\), hence the general solution is

\[y(x) = k_1 e^x + k_2 x e^x.\]

Use initial condition \(y(0) = 1\):

\[1 = k_1 e^0 + k_2 \cdot 0 \cdot e^0\]

hence \(k_1 = 1\) and consequently

\[y(x) = e^x + k_2 x e^x.\]
Example 4

\[
\begin{aligned}
&y'' - 2y' + y = 0, \\
y(0) = 1, \\
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\]

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\begin{cases}
  y'' - 2y' + y = 0, \\
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\end{cases}
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- The characteristic equation is
  \[\lambda^2 - 2\lambda + 1 = 0,\]
- The discriminant is 0, hence the general solution is
  \[y(x) = k_1 e^x + k_2 x e^x.\]
Example 4

\[
\begin{cases}
    y'' - 2y' + y = 0, \\
    y(0) = 1, \\
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\end{cases}
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- The characteristic equation is
  \[\lambda^2 - 2\lambda + 1 = 0,\]
- The discriminant is 0, hence the general solution is
  \[y(x) = k_1 e^x + k_2 x e^x.\]
- Use initial condition \(y(0) = 1:\)
  \[1 = y(0) = k_1 e^0 + k_2 \cdot 0 \cdot e^0\]
  hence \(k_1 = 1\) and consequently
  \[y(x) = e^x + k_2 x e^x.\]
Example 4, continued

\[ \begin{align*}
    y'' - 2y' + y &= 0, \\
y(0) &= 1, \\
y'(0) &= -1.
\end{align*} \Rightarrow y(x) = e^x + k_2 x e^x \]

Differentiate \( y(x) \):
\[ y'(x) = e^x + k_2 (e^x + x e^x) \]

Use \( y'(0) = -1 \):
\[ -1 = y'(0) = e^0 + k_2 (e^0 + 0) = 1 + k_2 \cdot 1 \]

Hence \( k_2 = -2 \) and consequently \( y(x) = e^x - 2x e^x \).
Example 4, continued

\[
\begin{aligned}
\left\{ 
\begin{array}{l}
y'' - 2y' + y = 0, \\
y(0) = 1, \\
y'(0) = -1.
\end{array}
\right.
\Rightarrow
\end{aligned}
\]

\[y(x) = e^x + k_2 x e^x\]

- Differentiate \(y(x)\):

\[y'(x) = e^x + k_2 \left(e^x + x e^x\right).\]
Example 4, continued

\[
\begin{cases}
y'' - 2y' + y = 0, \\
y(0) = 1, \\
y'(0) = -1.
\end{cases}
\]

\[
y(x) = e^x + k_2 x e^x
\]

- Differentiate \( y(x) \):

\[
y'(x) = e^x + k_2 (e^x + x e^x).
\]

- Use \( y'(0) = -1 \):

\[
-1 = y'(0) = e^0 + k_2 (e^0 + 0 \cdot e^0)
\]
Example 4, continued

\[
\begin{cases}
    y'' - 2y' + y = 0, \\
y(0) = 1, \\
y'(0) = -1.
\end{cases}
\implies y(x) = e^x + k_2 x e^x
\]

- Differentiate \(y(x)\):

\[
y'(x) = e^x + k_2 \left( e^x + x e^x \right).
\]

- Use \(y'(0) = -1\):

\[
-1 = y'(0) = e^0 + k_2 \left( e^0 + 0 \cdot e^0 \right)
\]
Example 4, continued

\[
\begin{aligned}
    y'' - 2y' + y &= 0, \\
y(0) &= 1, \\
y'(0) &= -1.
\end{aligned}
\Rightarrow y(x) = e^x + k_2 x e^x
\]

■ Differentiate \( y(x) \):

\[
y'(x) = e^x + k_2 \left( e^x + x e^x \right).
\]

■ Use \( y'(0) = -1 \):

\[
-1 = y'(0) = e^0 + k_2 \left( e^0 + 0 \cdot e^0 \right) = 1 + k_2.
\]

■ Hence \( k_2 = -2 \) and consequently

\[
y(x) = e^x - 2x e^x.
\]
General form of a boundary value problem:

\[
\begin{align*}
ay'' + by' + cy & = g(x), \\
y(x_1) & = y_1, \\
y(x_2) & = y_2.
\end{align*}
\]
General form of a boundary value problem:

\[
\begin{cases}
ay'' + by' + cy = g(x), \\
y(x_1) = y_1, \\
y(x_2) = y_2.
\end{cases}
\]

The conditions \(y(x_1) = y_1\) and \(y(x_2) = y_2\) are called boundary conditions.
General form of a boundary value problem:

\[
\begin{align*}
ay'' + by' + cy &= g(x), \\
y(x_1) &= y_1, \\
y(x_2) &= y_2.
\end{align*}
\] (2)

The conditions \( y(x_1) = y_1 \) and \( y(x_2) = y_2 \) are called boundary conditions.

Theorem

Sec 17.1, theorem 6

There is only one function \( y(x) \) that satisfies boundary value problem (2).
Example 5

\[
\begin{align*}
  y'' + 4y &= 0, \\
  y(0) &= 0, \\
  y\left(\frac{\pi}{12}\right) &= 1.
\end{align*}
\]

The characteristic equation is

\[\lambda^2 + 4 = 0,\]

The discriminant is negative, hence the general solution is

\[y(x) = k_1 \cos 2x + k_2 \sin 2x.\]

Use boundary condition \(y(0) = 0\):

\[0 = y(0) = k_1 \cdot 1 + k_2 \cdot 0 = k_1,\]

hence \(k_1 = 0\) and consequently

\[y(x) = k_2 \sin 2x.\]
Example 5

\[
\begin{align*}
y'' + 4y &= 0, \\
y(0) &= 0, \\
y\left(\frac{\pi}{12}\right) &= 1.
\end{align*}
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  \[\lambda^2 + 4 = 0,\]
Example 5

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\begin{aligned}
  y'' + 4y &= 0, \\
  y(0) &= 0, \\
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- The characteristic equation is
  \[\lambda^2 + 4 = 0,\]
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  \[y(x) = k_1 \cos 2x + k_2 \sin 2x.\]
Example 5

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\begin{aligned}
&y'' + 4y = 0, \\
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y\left(\frac{\pi}{12}\right) = 1.
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- The characteristic equation is
  \[\lambda^2 + 4 = 0,\]
- The discriminant is negative, hence the general solution is
  \[y(x) = k_1 \cos 2x + k_2 \sin 2x.\]
- Use boundary condition \(y(0) = 0:\)
  \[0 = y(0) = k_1 \cos 0 + k_2 \sin 0 = k_1 \cdot 1 + k_2 \cdot 0 = k_1\]
  hence \(k_1 = 0\) and consequently
  \[y(x) = k_2 \sin 2x.\]
Example 5, continued

\[
\begin{aligned}
  y'' + 4y &= 0, \\
y(0) &= 0, \\
y \left( \frac{\pi}{12} \right) &= 1.
\end{aligned}
\]

\[\Rightarrow \quad y(x) = k_2 \sin 2x\]
Example 5, continued

\begin{align*}
\begin{cases}
y'' + 4y = 0, \\
y(0) = 0, \\
y\left(\frac{\pi}{12}\right) = 1.
\end{cases} & \Rightarrow \\
y(x) = k_2 \sin 2x
\end{align*}

- Use condition \( y\left(\frac{\pi}{12}\right) = 1 \):

\[ 1 = y\left(\frac{\pi}{12}\right) = k_2 \sin \left(2 \cdot \frac{\pi}{12}\right) \]
Example 5, continued

\[
\begin{cases}
  y'' + 4y = 0, \\
y(0) = 0, \\
y\left(\frac{\pi}{12}\right) = 1.
\end{cases}
\Rightarrow
\]

\[y(x) = k_2 \sin 2x\]

- Use condition \(y\left(\frac{\pi}{12}\right) = 1:\)

\[1 = y\left(\frac{\pi}{12}\right) = k_2 \sin \left(2 \cdot \frac{\pi}{12}\right)\]
Example 5, continued

\[ \begin{align*}
\begin{cases}
    y'' + 4y &= 0, \\
y(0) &= 0, \\
y\left(\frac{\pi}{12}\right) &= 1.
\end{cases}
\end{align*} \]

\[ \Rightarrow \quad y(x) = k_2 \sin 2x \]

- Use condition \( y\left(\frac{\pi}{12}\right) = 1 \):

\[ 1 = y\left(\frac{\pi}{12}\right) = k_2 \sin \left(2 \cdot \frac{\pi}{12}\right) = k_2 \sin \left(\frac{\pi}{6}\right) \]
Example 5, continued

\[
\begin{align*}
\begin{cases}
  y'' + 4y &= 0, \\
  y(0) &= 0, \\
  y\left(\frac{\pi}{12}\right) &= 1.
\end{cases}
\end{align*}
\]

\[\Rightarrow\]

\[y(x) = k_2 \sin 2x\]

- Use condition \(y\left(\frac{\pi}{12}\right) = 1\):

\[1 = y\left(\frac{\pi}{12}\right) = k_2 \sin\left(2 \cdot \frac{\pi}{12}\right) = k_2 \sin\left(\frac{\pi}{6}\right) = \frac{1}{2} k_2.\]

- Hence \(k_2 = 2\) and consequently

\[y(x) = 2 \sin 2x.\]
2. Solve the following initial value problem:

\[
\begin{align*}
  y'' - 3y' &= 6, \\
  y(0) &= 2, \\
  y'(0) &= 1.
\end{align*}
\]
Newton’s second law:

*Mass times acceleration equals net force.*

Newton’s second law yields the following differential equation:

\[ mx''(t) + bx'(t) + kx(t) = F(t) \]
If $b = 0$ and $F(t) = 0$ then
\[ mx''(t) + kx(t) = 0. \]
If \( b = 0 \) and \( F(t) = 0 \) then

\[
mx''(t) + kx(t) = 0.
\]

The characteristic equation is

\[
m\lambda^2 + k = 0 \quad \implies \quad \lambda = \pm \sqrt{-\frac{k}{m}}
\]
The undriven MSD-system without damping behaves as an oscillator with frequency \( \omega = \sqrt{\frac{k}{m}} \).
If $b = 0$ and $F(t) = 0$ then

$$mx''(t) + kx(t) = 0.$$ 

The characteristic equation is

$$m\lambda^2 + k = 0 \quad \Rightarrow \quad \lambda = \pm \sqrt{-\frac{k}{m}}$$

Because $k, m > 0$ the general solution is

$$x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t), \quad \text{where} \quad \omega = \sqrt{\frac{k}{m}}.$$  

The undriven MSD-system without damping behaves as an oscillator with frequency $\omega = \sqrt{k/m}$. 


If \( F(t) = 0 \) then

\[
mx''(t) + bx'(t) + kx(t) = 0.
\]
If $F(t) = 0$ then

$$mx''(t) + bx'(t) + kx(t) = 0.$$ 

The characteristic equation is

$$m\lambda^2 + b\lambda + k = 0.$$ 

$$\lambda = -\frac{b}{2m} \pm \frac{1}{2m} \sqrt{b^2 - 4mk}$$
If $F(t) = 0$ then

$$mx''(t) + bx'(t) + kx(t) = 0.$$ 

The characteristic equation is

$$m\lambda^2 + b\lambda + k = 0.$$ 

$$\lambda = -\frac{b}{2m} \pm \frac{1}{2m} \sqrt{b^2 - 4mk}$$

A system is called **underdamped** if $b^2 - 4mk < 0$. In that case:

$$x(t) = e^{-bt/2m} \left[ c_1 \cos(\omega t) + c_2 \sin(\omega t) \right]$$

where

$$\omega = \frac{1}{2m} \sqrt{4mk - b^2}.$$
With \( F(t) = 0 \) the differential equation becomes

\[
mx''(t) + bx'(t) + kx(t) = 0.
\]
With $F(t) = 0$ the differential equation becomes

$$mx''(t) + bx'(t) + kx(t) = 0.$$ 

The characteristic equation is

$$m\lambda^2 + b\lambda + k = 0 \implies \lambda = -\frac{b}{2m} \pm \frac{1}{2m} \sqrt{b^2 - 4mk}.$$
With $F(t) = 0$ the differential equation becomes

$$mx''(t) + bx'(t) + kx(t) = 0.$$ 

The characteristic equation is

$$m\lambda^2 + b\lambda + k = 0 \implies \lambda = -\frac{b}{2m} \pm \frac{1}{2m} \sqrt{b^2 - 4mk}.$$ 

A system is called **critically damped** if $b^2 - 4mk = 0$. In that case:

$$x(t) = c_1 e^{-bt/2m} + c_2 te^{-bt/2m}. $$
With $F(t) = 0$ the differential equation becomes

$$mx''(t) + bx'(t) + kx(t) = 0.$$
With $F(t) = 0$ the differential equation becomes

$$mx''(t) + bx'(t) + kx(t) = 0.$$ 

The characteristic equation is

$$m\lambda^2 + b\lambda + k = 0 \quad \Rightarrow \quad \lambda = -\frac{b}{2m} \pm \frac{1}{2m} \sqrt{b^2 - 4mk}$$
With $F(t) = 0$ the differential equation becomes

$$mx''(t) + bx'(t) + kx(t) = 0.$$  

The characteristic equation is

$$m\lambda^2 + b\lambda + k = 0 \quad \Rightarrow \quad \lambda = -\frac{b}{2m} \pm \frac{1}{2m} \sqrt{b^2 - 4mk}$$

A system is called **overdamped** if $b^2 - 4mk > 0$. In that case:

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}.$$  

Geogebra applet: [damped spring system](#)
**Resistor:** limits the flow of current

**Inductor:** stores energy in a magnetic field

**Capacitor:** stores energy in an electric field
The voltage drops across the resistor, inductor, and capacitor are

\[ V_R(t) = R i(t), \quad V_L(t) = L i'(t), \quad V_C(t) = q(t)/C, \]

where \( q(t) \) is the charge on the capacitor.
The voltage drops across the resistor, inductor, and capacitor are

\[ V_R(t) = Ri(t), \quad V_L(t) = Li'(t), \quad V_C(t) = q(t)/C, \]

where \( q(t) \) is the charge on the capacitor.

Current equals charge per unit of time, so \( i(t) = q'(t) \).
The voltage drops across the resistor, inductor, and capacitor are

\[ V_R(t) = Ri(t), \quad V_L(t) = Li'(t), \quad V_C(t) = q(t)/C, \]

where \( q(t) \) is the charge on the capacitor.

Current equals charge per unit of time, so \( i(t) = q'(t) \).

Kirchhoff’s law \( V_R(t) + V_L(t) + V_C(t) = V_{ext}(t) \) implies

\[ Lq''(t) + Rq'(t) + \frac{1}{C} q(t) = V_{ext}(t) \]
RLC-circuits with no external voltage

- Differential equation for $V_{ext}(t) = 0$:

$$Lq''(t) + Rq'(t) + \frac{1}{C}q(t) = 0$$

The characteristic equation is

$$L\lambda^2 + R\lambda + \frac{1}{C} = 0.$$
RLC-circuits with no external voltage

- Differential equation for $V_{\text{ext}}(t) = 0$:

\[
Lq''(t) + Rq'(t) + \frac{1}{C}q(t) = 0
\]

- The characteristic equation is

\[
L\lambda^2 + R\lambda + \frac{1}{C} = 0.
\]
RLC-circuits with no external voltage

Differential equation for $V_{ext}(t) = 0$:

$$Lq''(t) + Rq'(t) + \frac{1}{C}q(t) = 0$$

The characteristic equation is

$$L\lambda^2 + R\lambda + \frac{1}{C} = 0.$$ 

The solutions are similar to the MSD case:

- underdamped solutions for $R^2 - 4L/C < 0$,
- critically damped solutions for $R^2 - 4L/C = 0$,
- overdamped solutions for $R^2 - 4L/C > 0$. 
Assignment: IMM1 - Tutorial 3.4